

# Mass and Weak Field Limit of Boson Stars in Brans Dicke Gravity

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We study boson stars in Brans Dicke gravity and use them to illustrate some of the properties of three different mass definitions: the Schwarzschild mass, the Keplerian mass and the Tensor mass. We analyse the weak field limit of the solutions and show that only the Tensor mass leads to a physically reasonable definition of the binding energy. We examine numerically strong field  $\omega = -1$  solutions and show how, in this extreme case, the three mass values and the conserved particle number behave as a function of the central boson field amplitude. The numerical studies imply that for  $\omega = -1$ , solutions with extremal Tensor mass also have extremal particle number. This is a property that a physically reasonable definition of the mass of a boson star must have, and we prove analytically that this is true for all values of  $\omega$ . The analysis supports the conjecture that the Tensor mass uniquely describes the total energy of an asymptotically flat solution in BD gravity.

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## I. INTRODUCTION

Boson stars (or Klein Gordon geons) are asymptotically flat self gravitating configurations of zero temperature scalar particles (bosons). The boson field is described by a complex wave function  $\Psi$  and the matter Lagrangian possesses a  $U(1)$  internal symmetry. This symmetry leads to the existence of a conserved charge  $N$  which is interpreted as the total number of bosons. The first boson star solutions were found by Kaup [1] and independently by Ruffini & Bonazzola [2]. These authors examined spherically symmetric boson stars in General Relativity (GR) for which the only self interaction term in the matter Lagrangian is the boson mass. They found that the solutions exhibit qualitatively similar behaviour to those for neutron stars and white dwarfs: they form a single parameter family whose ADM mass  $M_{ADM}$  and particle number  $N$  vary smoothly with the parameter and have coinciding maxima and minima. However, the ADM masses of the stars were found to be extremely small, of order  $M_{pl}^2/\alpha$ , where  $M_{pl} := \sqrt{\hbar/G}$  is the Planck mass and  $\alpha$  is the mass of the bosons. Colpi, Shapiro and Wasserman [3] studied GR boson stars whose matter Lagrangian includes a quartic self interaction term. They found that this term dominates the pressure of the stars and increases the ADM mass to order  $M_{pl}^3/\alpha^2$ .

The stability of boson stars in GR has been studied by, amongst others, Kusmartsev, Mielke & Schunck [4]- [5], who used an approach based on catastrophe theory. To use this method one must identify two conserved charges that vary smoothly with some parameter which labels the solutions and whose extrema occur at the same values of this parameter. Then a plot of one charge against the other will show cusps at the extremal points and the first of these cusps marks the onset of dynamical instability. For GR boson stars, the appropriate conserved charges are  $N$  and  $M_{ADM}$ , and Jetzer [6] has proved that these two quantities do have coinciding extrema both for pure boson stars and for mixed boson-fermion stars.

There has been renewed interest recently in generalising GR by adding one or more scalar (dilaton) gravitational fields. These extra fields appear in the low energy limits of super string and super-gravity theories. The simplest low energy string theory effective action is approximated by the  $\omega = -1$  action of Brans Dicke (BD) theory [7], which includes one dilaton field. This choice of BD parameter has been ruled out by solar system experiments that imply that  $\omega \geq 500$ . However, these experiments do not place such strong limits on more general Scalar Tensor (ST) theories in which the coupling parameter  $\omega$  is allowed to vary. Thus it is conceivable that the gravitational interaction can

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be described by a ST theory in which  $\omega \approx -1$  at some early time when string theory is still valid, and that this parameter has increased to its present value as the Universe has evolved to its present state.

In the original (Jordan frame) formulation of BD theory, Lee [8] has shown that  $M_{ADM}$  is not a good description of the total energy of a spacetime. In particular, it is not conserved by an isolated source that emits gravitational radiation. To obtain a conserved mass one must modify  $M_{ADM}$  by adding another term that involves the derivative of the dilaton field.

Several authors have studied boson stars within the framework of ST gravity. Gunderson & Jensen [9] considered boson stars in BD theory, focusing on  $\omega = 6$  solutions, and they showed that these solutions are qualitatively similar to those in GR. Tao & Xue [10] discuss solutions to a scale invariant variation of BD theory in which the mass term in the matter Lagrangian couples to the curvature via the dilaton. This theory necessarily violates the Weak Equivalence Principle and the authors found a conserved charge associated with both the boson and dilaton fields. Torres [11] examined boson stars in more general ST theories while Comer & Shinkai [12] used catastrophe theory to examine the stability of ST boson stars.

When one considers a ST boson star in a cosmological setting, the evolution of the background (cosmological) value of the dilaton field may differ from the evolution of the dilaton field near to the centre of the boson star. This phenomena was originally described by Barrow as “gravitational memory” [13], although he considered the effect of cosmological evolution on the horizon area of a black hole. However, as pointed out by Torres, Liddle & Schunck [14], ST boson stars also offer a means of testing theoretically the degree to which the gravitational memory hypothesis is true, since boson stars are also highly relativistic objects and have the added advantage of being singularity free. These authors have also studied the gravitational evolution of a BD boson star in a cosmological background [15] and they found that a star that is stable at the current epoch remains stable as one traces its evolution backwards in cosmological time.

In this paper we examine how one defines mass in ST gravity and use boson star solutions to illustrate the properties of three mass definitions: the Schwarzschild mass (which gives the ADM mass in the asymptotic limit), the Keplerian mass (that quantity that determines the geodesic motion of non-self gravitating test particles) and the Tensor mass (the conserved quantity identified by Lee [8]). For the sake of simplicity we consider only BD theory. In the spirit of the original formulation of the theory we assume that the Jordan frame is the physical frame. We focus on  $\omega = -1$  strong field solutions since in this case the field equations are sufficiently different to the GR equations for strong field scalar-tensor effects to appear. Also, as mentioned above, this value of the coupling parameter may be relevant to the study of boson stars in the early Universe. However, our analytical results are valid for any value of  $\omega$ .

The plan of the paper is as follows. In Section II we give the field equations and discuss the boundary conditions that must be imposed to obtain boson star solutions. In Section III we discuss three definitions of quasi-local mass adapted to the symmetries of the solutions and give an expression for a generalisation of the ADM mass that is valid in BD theory. In Section IV we examine the asymptotic form of the solutions and show that in the asymptotic limit, the three quasi-local masses each tend towards one of the generalised ADM masses. In Section V we describe how the physical characteristics of the solutions change under a rescaling of the dilaton field. In Section VI we discuss the weak field limit of the solutions and derive an expression for the fractional binding energy in this limit. In Section VII we describe some of the properties of the strong field solutions, focusing on the  $\omega = -1$  coupling and highlighting the differences between the three mass values we are considering. In Section VIII we prove that for all  $\omega$ , solutions of extremal particle number are those of extremal Tensor mass. Finally, in Section IX we give some concluding remarks. Throughout this paper we use the sign and index conventions of [16].

## II. STRUCTURE EQUATIONS AND BOUNDARY CONDITIONS

We take as our starting point the action for Brans-Dicke gravity minimally coupled to a complex boson field  $\Psi$ :

$$S = \int d^4x \sqrt{-g} \left[ \frac{e^{-\phi}}{16\pi G} (\mathcal{R} - \omega \nabla^\sigma \phi \nabla_\sigma \phi) - \frac{1}{4} (\nabla^\sigma \Psi^* \nabla_\sigma \Psi + \nabla^\sigma \Psi \nabla_\sigma \Psi^*) - \frac{\alpha^2}{2\hbar^2} \Psi^* \Psi \right] \quad (1)$$

where  $\alpha$  is the mass of the boson particles and we have chosen units in which  $c = 1$ . The dilaton field  $\phi$  is dimensionless and the product  $e^\phi G$  measures the strength of the gravitational coupling. Varying the metric leads to the field equations

$$G_{\mu\nu} = (1 + \omega) \nabla_\mu \phi \nabla_\nu \phi - (1 + \omega/2) g_{\mu\nu} \nabla^\sigma \phi \nabla_\sigma \phi - \nabla_\mu \nabla_\nu \phi + g_{\mu\nu} \square \phi + 8\pi G e^\phi T_{\mu\nu} \quad (2)$$

where

$$T_{\mu\nu} = \frac{1}{2} \left( \nabla_\mu \Psi^* \nabla_\nu \Psi + \nabla_\mu \Psi \nabla_\nu \Psi^* - g_{\mu\nu} \frac{\alpha^2}{\hbar^2} \Psi^* \Psi \right) - \frac{1}{4} g_{\mu\nu} (\nabla^\sigma \Psi^* \nabla_\sigma \Psi + \nabla^\sigma \Psi \nabla_\sigma \Psi^*) \quad (3)$$

is the energy momentum tensor of the boson field. Varying the dilaton and boson fields gives the dilaton wave equation

$$2\omega \square \phi = \mathcal{R} + \omega \nabla^\sigma \phi \nabla_\sigma \phi \quad (4)$$

and the boson wave equations

$$\square \Psi = \frac{\alpha^2}{\hbar^2} \Psi, \quad \square \Psi^* = \frac{\alpha^2}{\hbar^2} \Psi^*. \quad (5)$$

The action possesses a global  $U(1)$  symmetry which implies the existence of a conserved current

$$J^\mu = \frac{i}{2} g^{\mu\nu} (\Psi^* \nabla_\nu \Psi - \Psi \nabla_\nu \Psi^*), \quad \nabla_\mu J^\mu = 0. \quad (6)$$

Note that  $J^\mu$  has no explicit dependence on  $\phi$ : eqn (6) is identical to the GR definition since the boson field is minimally coupled to  $\mathcal{R}$ . For an asymptotically flat spacetime and time-like current, eqn (6) gives a conserved (time-independent) charge

$$N = \int d^3x \sqrt{h} n_\mu J^\mu \quad (7)$$

where the integral is taken over an arbitrary space-like hypersurface with time-like unit normal  $n_\mu$  and induced metric  $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ . In contrast with the solutions discussed in [10], there is no conserved charge associated with the dilaton field. Although we have written the dilaton terms on the right of eqn (2), so that they contribute to the total energy momentum tensor, we interpret  $\phi$  as an extra component of the gravitational field, not as an additional particle field.

We consider static, spherically symmetric equilibrium solutions. Using the standard orthogonal  $\{t, r, \theta, \varphi\}$  coordinate basis we write the line element as

$$ds^2 = -e^{2\nu} dt^2 + e^{2\lambda} dR^2 + R^2 d\theta^2 + R^2 \sin^2 \theta d\varphi^2 \quad (8)$$

where  $\nu$  and  $\lambda$  are functions of  $R$ . We denote the normalised time-like Killing vector field by  $\xi^\mu = (e^{-\nu}, 0, 0, 0)$ . Our solutions will be singularity and horizon free, so the coordinate system defined by eqn (8) is well behaved. We look for solutions of minimum energy. One can show [17] that the form of  $\Psi$  compatible with this requirement is given by

$$\Psi = \frac{P}{\sqrt{8\pi G}} \exp \left( i \frac{\alpha \Omega}{\hbar} t \right), \quad (9)$$

where  $\Omega$  is a real dimensionless constant and  $P$  is a real dimensionless function of  $R$ . A further requirement is that the eigenfunction  $P(R)$  possesses no nodes, so that the bosons are in their ground state. Then the product  $\Omega\alpha$  defines the ground-state energy of the bosons in the zero node eigenstate  $P(R)$ . From the line element (8) and the wave-function (9), it is easy to show that  $T_{\mu\nu}$  obeys the weak energy condition.

Before writing the equations of motion, we define a new dimensionless radial coordinate

$$r := \frac{\alpha R}{\hbar}. \quad (10)$$

Then, using eqns (8) and (9), the independent components of the field and wave equations reduce to the following system of coupled ODEs:

$$\nu' = \frac{1}{2\phi' r - 4} \left[ e^{2\lambda+\phi} P^2 r (1 - \Omega^2 e^{-2\nu}) - e^\phi P'^2 r - \phi'^2 r \omega - 4\phi' - \frac{2}{r} (e^{2\lambda} - 1) \right], \quad (11)$$

$$\begin{aligned} \lambda' = \frac{e^\phi r}{4(2\omega + 3)} & \left[ e^{2\lambda} P^2 (\Omega^2 e^{-2\nu} (2\omega + 5) + 2\omega - 1) + P'^2 (2\omega + 1) \right] \\ & + \frac{\phi'^2 r \omega}{4} + \frac{\phi' \nu' r}{2} - \frac{1}{2r} (e^{2\lambda} - 1), \end{aligned} \quad (12)$$

$$\phi'' = \frac{e^\phi}{2\omega + 3} [e^{2\lambda} P^2 (2 - \Omega^2 e^{-2\nu}) + P'^2] + \phi'^2 + \phi' \left( \lambda' - \nu' - \frac{2}{r} \right) \quad (13)$$

and

$$P'' = e^{2\lambda} P (1 - \Omega^2 e^{-2\nu}) + P' \left( \lambda' - \nu' - \frac{2}{r} \right), \quad (14)$$

where the prime denotes  $\frac{d}{dr}$ . To solve these equations we impose the boundary conditions

$$e^{\lambda_0} = 1, \quad P'_0 = 0, \quad \phi'_0 = 0, \quad P_\infty = 0 \quad (15)$$

where throughout this paper the subscripts '0' and ' $\infty$ ' denote values at  $r = 0$  and  $r = \infty$  respectively. The first three conditions enforce regularity of the solutions at the origin, while the last condition ensures that the boson matter is localised and the solutions are asymptotically flat. For any given  $\omega$  the solutions may be parameterised by  $P_0$  and  $\phi_\infty$ . The field equations then become eigenvalue equations for  $\Omega$ . Note that the value of  $e^\nu$  at infinity is arbitrary: eqns (11) to (14) are invariant under the rescaling  $\nu \rightarrow \nu + k$ ,  $\Omega \rightarrow \Omega e^k$  where  $k$  is constant. We use this freedom to set

$$e^{\nu_\infty} = 1 \quad (16)$$

which fixes the scale of our time coordinate and consequently our unit of energy. The field equations automatically lead to the asymptotic conditions

$$P'_\infty = \nu'_\infty = \phi'_\infty = \lambda_\infty = 0. \quad (17)$$

From eqns (6) and (9),  $J^\mu$  is parallel to  $\xi^\mu$  and has the non-zero component

$$J^0 = \frac{\hbar \Omega P^2}{8\pi \alpha G} e^{-2\nu}. \quad (18)$$

Choosing a hypersurface orthogonal to  $\xi^\mu$  and combining the above expression with eqn (7) we have

$$N = \frac{1}{2} \int_0^\infty e^{\lambda-\nu} \Omega P^2 r^2 dr \quad (19)$$

where we have expressed  $N$  in units of  $M_{pl}^2/\alpha = \hbar/\alpha G$ . Since there are no Maxwell terms present in the action (1), the bosons have no electromagnetic charge and we interpret the quantity  $\frac{1}{\alpha} N$  as the total boson particle number so that  $N$  is the total rest mass of the star. This latter quantity is simply the sum of the masses of the bosons that make up the star, as measured by some non-gravitational experiment. From the rest mass we define the Newtonian mass

$$M_N := N e^{\phi_\infty} \quad (20)$$

which measures the gravitational mass (or energy) of a star whose bosons are dispersed to infinity.

Finally we note that to recover the GR equations of motion and their corresponding solutions we take the limit  $\omega \rightarrow \infty$  which implies that  $\phi \rightarrow \phi_{GR}$ , where  $\phi_{GR}$  is constant. Since we have included Newton's constant  $G$  in the action (1), we have used up the freedom to scale the dilaton (which is equivalent to rescaling the unit of mass) and so we have the additional requirement  $\phi_{GR} = 0$ .

### III. QUASI-LOCAL AND ADM MASSES

Operationally, in Newtonian theory one determines the active gravitational mass  $GM$  of a gravitating source by applying Kepler's third law. A test particle in a circular orbit of radius  $R$  and angular velocity  $d\varphi/dt$  about the source measures an active mass

$$GM = \lim_{R \rightarrow \infty} \left[ R^3 \left( \frac{d\varphi}{dt} \right)^2 \right]. \quad (21)$$

This mass is a product of the gravitational coupling strength  $G$  and the rest mass  $M$  of the source. If the source is spherically symmetric, eqn (21) is valid for all  $R$  and it measures the gravitational mass  $GM(R)$  contained within the 2-sphere  $\Sigma(R)$  of radius  $R$ .

For metric theories of gravity the determination of mass is more difficult. All we can discuss is some total gravitational mass, which is determined by the metric  $g_{\mu\nu}$  and any additional tensor or scalar gravitational fields appearing in the theory. Furthermore, for metric theories other than GR, the Strong Equivalence Principle (SEP, defined in [18]) is violated. Because of this, the mass we measure using Kepler's third law depends upon the test particle's gravitational binding energy. Hence, in an alternative theory of gravity we cannot in general separate the gravitational coupling strength from the rest mass of the source.

Returning to the boson star solutions, we first discuss three definitions of quasi-local mass (QLM), adapted to the symmetries of the line element (8). As is well known, QLM definitions are of limited utility. One reason for this is that it is difficult to write a coordinate invariant conservation law for any quasi-local quantity. However, in a static spacetime, the time-like Killing vector field  $\xi^\mu$  provides a natural splitting of the spacetime into a set of space-like hypersurfaces of constant time on which one can formulate an expression for the QLM. The existence of  $\xi^\mu$  guarantees that quantities defined on the constant time slices are conserved. There are still an infinite number of ways of defining QLM in a static spacetime. However, an acceptable definition of the mass  $M(\mathcal{U})$  of a sub-manifold  $\mathcal{U}$  must satisfy the following three requirements: (a)  $M$  is non-negative, (b)  $M = 0$  everywhere in Minkowski spacetime, and (c) given two sub-manifolds  $\mathcal{U}$  and  $\mathcal{V}$ , such that  $\mathcal{U} \subset \mathcal{V}$ , then  $M(\mathcal{U}) \leq M(\mathcal{V})$ . We also feel that any global mass defined as the limit of some QLM is only valid if the QLM satisfies these conditions, and we use this criterion to select physically reasonable definitions of mass for the static boson star solutions we consider here.

We first consider the generalised Schwarzschild mass, which is given by any one of the familiar definitions

$$m_S(r) = \frac{r}{2} (1 - e^{-2\lambda}) = \int_0^r \frac{\tilde{r}^2}{2} \rho d\tilde{r} = \int_0^r \left[ \frac{1}{2} (1 - e^{-2\lambda}) + \lambda' \tilde{r} e^{-2\lambda} \right] d\tilde{r} \quad (22)$$

where  $m_S(r)$  is measured in units of  $M_{pl}^2/\alpha$  and  $\rho$  is the total energy density defined by

$$\rho := G_{\mu\nu} \xi^\mu \xi^\nu. \quad (23)$$

Equation (22) is the definition of mass adopted in all references on boson stars cited in the introduction except [12], and it gives an unambiguous definition of the energy of a spherically symmetric, asymptotically flat system in GR (see [16], p. 603): provided  $\rho$  is non-negative,  $m_S$  satisfies conditions (a) to (c) outlined above and in the limit  $r \rightarrow \infty$ ,  $m_S$  tends to the ADM mass. However, in Brans Dicke theory one or more of these conditions may be violated even for a physically reasonable energy momentum tensor. From eqns (2) and (23) the total energy density of the boson star is given by

$$\rho = e^{-2\nu} G_{00} = \left(1 + \frac{\omega}{2}\right) e^{-2\lambda} \phi'^2 - \Delta\phi + \frac{e^\phi}{2} (\Omega^2 P^2 e^{-2\nu} + P^2 + P'^2 e^{-2\lambda}) \quad (24)$$

where

$$\Delta\phi := \frac{e^{-\lambda}}{r^2} \frac{d}{dr} (r^2 e^{-\lambda} \phi'). \quad (25)$$

This expression contains non-positive definite terms so although  $T_{\mu\nu}$  obeys the weak energy condition the total density may still be negative in some regions of a solution, leading to the condition  $m'_S < 0$  (in violation of requirement (c)). This is indeed the case for the  $\omega = -1$  solutions discussed later, and there even exist weak field solutions for which  $m_S < 0$  everywhere. The fact that the QLM  $m_S$  may have a negative gradient leads one to conclude that the ADM mass, which as we shall see below is given by  $\lim_{R \rightarrow \infty} m_S$ , fails to account for all of the energy of the spacetime.

As an alternative to eqn (22), we may determine the gravitational mass enclosed within a two sphere  $\Sigma(r)$  by measuring the analogue of the gravitational acceleration of a test particle at radius  $r$ . More specifically, we consider circular geodesic orbits and use eqn (21) to determine the mass enclosed by the orbit. This definition of mass is valid in a static, spherically symmetric spacetime. In this case we may project a particle orbit of constant  $r$  onto a hypersurface orthogonal to  $\xi^\mu$  to form a circle about the centre of symmetry. The set of all orbits at this radius, projected in the same way, then form a closed two sphere  $\Sigma(r)$ . The matter and gravitational fields contained within this sphere remain constant in time, and so the orbital mass is well defined. Since the SEP is violated in BD theory, the value of the orbital mass we measure depends upon the gravitational binding energy of the orbiting test particle.

We consider two orbital masses: the Keplerian and the Tensor mass. A *non-self gravitating* test particle in a circular geodesic orbit in the geometry of eqn (8) moves with an angular velocity

$$\frac{d\varphi}{dt} = e^\nu \sqrt{\frac{\nu'}{r}} \quad (26)$$

as measured by an observer at infinity. Use of eqn (21) then leads to the Keplerian mass function

$$m_K(r) = \nu' r^2 e^{2\nu} \quad (27)$$

in units of  $M_{pl}^2/\alpha$ . The Tensor mass we define similarly by considering the circular orbit of a test particle in the Einstein frame, whose metric  $\tilde{g}_{\mu\nu}$  is conformally related to the physical (Jordan frame) metric  $g_{\mu\nu}$  by the transformation  $\tilde{g}_{\mu\nu} = e^{-\phi} g_{\mu\nu}$ . The orbital angular velocity in the Einstein frame is given by

$$\frac{d\varphi}{dt} = e^\nu \sqrt{\frac{2\nu' - \phi'}{r(2 - r\phi')}} \quad (28)$$

Kepler's third law (21) then gives the Tensor mass function

$$m_T(r) = \frac{r^2 e^{2\nu} (2\nu' - \phi')}{2 - \phi' r}, \quad (29)$$

which may be interpreted as the mass measured by an orbiting test black hole in the Jordan frame [19]. As with the previous mass functions, the Tensor mass is given in units of  $M_{pl}^2/\alpha$ . Note that the conformal transformation used to define the Tensor mass is merely a computational device. We use it to find a frame in which the test particle's motion is geodesic and use this motion to define the Tensor mass. We could equally well define a different orbital mass by transforming to a different conformal frame.

Numerical calculations for the  $\omega = -1$  BD solutions show that both  $m_K$  and  $m_T$  are non-decreasing functions of  $r$ . From eqns (26) and (28), both mass functions are also non-negative definite (as a corollary eqn (27) implies that  $e^\nu$  is a non-decreasing function of  $r$ ). In Minkowski spacetime one has  $m_K = m_T = 0$  everywhere. Hence the Keplerian and Tensor masses satisfy conditions (a) and (b) given above and, at least for the  $\omega = -1$  solutions studied later, they appear to satisfy condition (c).

Lee [8] has shown that one can derive ADM-like masses in Brans-Dicke gravity from the superpotential

$$H^{\mu[\nu\alpha]\beta} = (-g)e^{n(\phi_\infty - \phi)} (g^{\mu\nu} g^{\alpha\beta} - g^{\mu\alpha} g^{\nu\beta}) \quad (30)$$

where  $n$  is usually taken to be an integer. Note that this expression is not unique due to the ambiguity in the choice of pseudo-tensor for the gravitational field. Furthermore, the index  $n$  is undetermined, which reflects the additional ambiguity we have in determining the contribution the dilaton field makes to the total gravitational energy momentum.

The potential satisfies the conservation law  $H^{\mu\nu\alpha\beta}_{,\nu\alpha\beta} = 0$  and on integrating this expression and using Stoke's theorem twice we have the generalised ADM mass

$$M_n = -\frac{1}{16\pi} \int \left[ (-g)e^{n(\phi_\infty - \phi)} (g^{00} g^{ij} - g^{0i} g^{0j}) \right]_{,j} d^2 \Sigma_i \quad (31)$$

where the integral is performed over the 2-sphere  $\Sigma(r)$  in the limit  $r \rightarrow \infty$  and the metric must be expressed in rectangular (Minkowski) coordinates. Our definition of  $H^{\mu\nu\alpha\beta}$  differs from the one given in [8] by a factor of  $e^{n\phi_\infty}$  which is included here so that  $M_n$  is expressed in units of  $M_{pl}^2/\alpha$ . For the metric defined by eqn (8), the integral (31) evaluates to

$$M_n = \lim_{r \rightarrow \infty} \left[ r (1 - e^{-\lambda}) \right] + \frac{n\phi_1}{4} \quad (32)$$

where

$$\phi_1 = \lim_{r \rightarrow \infty} (r^2 \phi'). \quad (33)$$

In the following Section we shall relate this expression to the asymptotic limits of the quasi-local masses.

#### IV. THE ASYMPTOTIC FORM OF THE SOLUTIONS

Expanding the metric and mass functions in powers of  $1/r$  about  $r = \infty$ , one can show that to order  $1/r$  the line element is given by

$$ds^2 = - \left(1 - \frac{2M_K}{r}\right) dt^2 + \left(1 + \frac{2(M_K - \phi_1)}{r}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2. \quad (34)$$

where  $M_K$  is the limit of eqn (27) as  $r \rightarrow \infty$  and  $\phi_1$  is defined in eqn (33). Since  $\lim_{r \rightarrow 0}(r^2 \phi') = 0$  this quantity may also be written as

$$\phi_1 = \int_0^\infty \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) dr. \quad (35)$$

Equation (34) is a special case of the vacuum BD solution given in [7] and the asymptotic solution is determined by two parameters which we take here to be  $\phi_1$  and  $M_K$ . For this form of the metric, the boson wave equation (14) has the asymptotic solution

$$P = r^b e^{-\kappa r} \left[ 1 + \mathcal{O}\left(\frac{1}{r}\right) \right] \quad (36)$$

where

$$b = -1 - \kappa(M_K - \phi_1) + \frac{\Omega^2}{\kappa} M_K, \quad \kappa = \sqrt{1 - \Omega^2}. \quad (37)$$

The boson field falls off exponentially with  $r$ , far more rapidly than any other field. This justifies the use of the name “star” in describing these solutions: although the object has no well defined surface, the boson matter is still highly localised. The parameter  $\kappa$  may be interpreted as the reciprocal radius of the star and must be real for an acceptable solution. This implies that  $\Omega \leq 1$ .

These solutions are asymptotically flat so that for large  $r$  we have  $1 - e^{-\lambda} \sim \frac{1}{2}(1 - e^{-2\lambda})$ . Hence, setting  $n = 0$  in eqn (32) and comparing the result with eqn (22) gives

$$M_0 = M_{ADM} = \lim_{r \rightarrow \infty} m_S \quad (38)$$

and we see that, as mentioned above, the generalised Schwarzschild mass tends to the ADM mass in the Jordan frame. Taking the leading terms in the expansion of the Keplerian and Tensor masses, one can show that

$$M_K = M_4 = M_{ADM} + \phi_1, \quad M_T = M_2 = M_{ADM} + \frac{\phi_1}{2}. \quad (39)$$

Note that  $M_T$  is the ADM mass of the star in the Einstein frame, which is not equal to  $M_{ADM}$  since the derivative of the conformal factor  $e^{-\phi}$  which relates the two frames is non-vanishing. Comer and Shinkai [12] use the correct definition of mass (the Schwarzschild mass in the Einstein frame) but, as we have seen, this quantity is not the same as the Jordan frame ADM mass (this contradicts a statement given in [12]). Numerical studies show that  $\phi' > 0$  for all  $\omega = -1$  solutions investigated. This implies that

$$M_{ADM} \leq M_T \leq M_K \quad (40)$$

for these solutions.

For each value of the mass  $M_n$  there is an associated fractional binding energy  $E_n$  per unit boson mass given by

$$E_n = (M_n - M_N)/M_N. \quad (41)$$

This expression is independent of the unit of mass chosen. The choice of which  $E_n$  correctly measures the binding energy of the star depends upon which mass value  $M_n$  corresponds to the true energy of the star.

## V. DILATON RESCALING

As mentioned above, the solutions may be parameterised by the pair  $P_0, \phi_\infty$ . The former quantity determines the energy density of the boson field at the origin and for each pair of parameters  $(P_0, \phi_\infty)$  there is a unique value of pair of  $\Omega e^{-\nu_0}$  that gives a zero node solution with the boundary conditions (15) and (16). The product  $\Omega e^{-\nu_0}$  measures the bosons' ground state energy per unit boson mass at the origin, as seen by an observer at  $r = \infty$ . This quantity increases monotonically with  $P_0$  and the pair  $(\Omega e^{-\nu_0}, \phi_\infty)$  serves as an alternative parameterisation of the solutions. Note that  $\Omega e^{-\nu_0}$  is invariant under the rescaling of  $e^\nu$  described before eqn (16).

The equations of motion and the mass equations (20), (22), (27), (29) and (32) are invariant under the rescaling

$$e^\phi \rightarrow k^2 e^\phi, \quad P \rightarrow \frac{P}{k} \quad (42)$$

where  $k$  is a constant, while  $N$  re-scales under eqn (42) as

$$N \rightarrow \frac{N}{k^2}. \quad (43)$$

This rescaling swaps energy between the dilaton part of the gravitational field and the boson energy density in such a way as to leave the total gravitating energy invariant. Given a set of solutions

$$\mathcal{S}(P_0; \phi_\infty) = \{M(P_0; \phi_\infty), N(P_0; \phi_\infty)\} \quad (44)$$

parameterised by  $P_0$  for some fixed  $\phi_\infty$ , where  $M$  is any asymptotic or quasi-local mass value, eqn (42) generates a new physically distinct set

$$\mathcal{S}(P_0; \phi_\infty + 2 \log k) = \left\{ M(kP_0; \phi_\infty), \frac{1}{k^2} N(kP_0; \phi_\infty) \right\}. \quad (45)$$

This new set of solutions have exactly the same mass and binding energy as the first set, but consists of stars of different total particle numbers with compensating changes in their gravitational coupling strength. Hence, eqn (45) may be used to compare boson star solutions in cosmological settings with different values of  $\phi_\infty$ .

## VI. WEAK FIELD SOLUTIONS

In this section we consider the behaviour of the zero node solutions in the weak field limit, which we define as the limit in which  $P(r)$  is small but non-zero. Following Kaup [20], we write the boson field amplitude as

$$P = \varepsilon \Pi(a) \quad (46)$$

where  $0 < \varepsilon \ll 1$  and  $a$  is a new radial coordinate defined by  $a = \sqrt{\varepsilon} r$ . The function  $\Pi$  is defined to have the boundary value  $\Pi_0 = 1$ , so that  $\varepsilon$  measures the degree to which the solutions differ from flat spacetime. We use  $\varepsilon$  as an expansion parameter and to first order in  $\varepsilon$  we expand the metric functions, dilaton field and energy eigenvalue about flat spacetime as

$$\lambda = \frac{\varepsilon}{2} A(a) \quad \nu = \frac{\varepsilon}{2} B(a) \quad \phi = \phi_\infty + \varepsilon \Phi(a) \quad \Omega = 1 - \frac{\varepsilon}{2} \Gamma \quad (47)$$

where  $\Gamma(P_0, \phi_\infty)$  is positive and constant for each solution. Equations (11) to (14) then reduce to

$$(a^2 B')' = \frac{2(\omega + 2)}{2\omega + 3} a^2 \Pi^2 e^{\phi_\infty} + \mathcal{O}(\varepsilon), \quad (48)$$

$$(aA)' = \frac{2(\omega + 1)}{2\omega + 3} a^2 \Pi^2 e^{\phi_\infty} + \mathcal{O}(\varepsilon), \quad (49)$$

$$(a^2 \Phi')' = \frac{a^2 \Pi^2 e^{\phi_\infty}}{2\omega + 3} + \mathcal{O}(\varepsilon) \quad (50)$$



and

$$(a^2\Pi')' = a^2\Pi(B + \Gamma) + \mathcal{O}(\varepsilon) \quad (51)$$

where throughout this section a prime denotes  $\frac{d}{da}$ .

Using eqns (19), (20), (22) and (24), the derivatives of eqns (27) and (29), and the weak field equations of motion the quantities  $M_N$ ,  $M_{ADM}$ ,  $M_T$  and  $M_K$  may be written as the integrals

$$M_N = e^{\phi_\infty} N = \sqrt{\varepsilon} \int_0^\infty \frac{a^2 \Pi^2 e^{\phi_\infty}}{2} da + \mathcal{O}(\varepsilon^{3/2}) \quad (52)$$

$$M_{ADM} = M_0 = \sqrt{\varepsilon} \int_0^\infty a^2 \Pi^2 e^{\phi_\infty} \frac{(\omega + 1)}{2\omega + 3} da + \mathcal{O}(\varepsilon^{3/2}) \quad (53)$$

$$M_T = M_2 = \sqrt{\varepsilon} \int_0^\infty \frac{a^2 \Pi^2 e^{\phi_\infty}}{2} da + \mathcal{O}(\varepsilon^{3/2}) \quad (54)$$

$$M_K = M_4 = \sqrt{\varepsilon} \int_0^\infty a^2 \Pi^2 e^{\phi_\infty} \frac{(\omega + 2)}{2\omega + 3} da + \mathcal{O}(\varepsilon^{3/2}). \quad (55)$$

Equation (53) shows that, in the weak field limit, the ADM mass is negative for  $\omega < -1 - \varepsilon$ . This is a consequence of treating  $\phi$  solely as a matter field. The energy density may be written partly in terms of the weak field variables to give

$$\rho = -\frac{\varepsilon^2}{a^2} (a^2 \Phi')' + 8\pi G e^\phi T_{\mu\nu} \xi^\mu \xi^\nu, \quad (56)$$

where we have shown the dilaton contribution to lowest non-zero order in  $\varepsilon$  and one can show that the second term on the right is also of order  $\varepsilon^2$ . Equation (50) implies that  $(a^2 \Phi')'$  is positive, so that the dilaton contribution to the total density in eqn (56) is negative. As  $\omega$  decreases below  $\omega \sim -1$ , the derivatives of the dilaton field increase in magnitude and the total density becomes dominated by the (negative) dilaton term. This leads to a negative Schwarzschild mass gradient for all  $r$  and so gives a negative quasi-local mass  $m_S$ , and hence negative  $M_{ADM}$ . When  $\omega > -1 + \varepsilon$  the coupling between the dilaton and the curvature is weak enough for the boson field density to dominate over the dilaton density, so that in this case both  $m_S$  and  $M_{ADM}$  are positive. For  $|\omega - 1| \sim \varepsilon$  we need to go to higher order in  $\varepsilon$  to calculate  $M_{ADM}$ . However, numerical calculations show that  $M_{ADM}$  increases smoothly from negative to positive values as  $\omega$  increases over the interval  $(-1 - \varepsilon, -1 + \varepsilon)$ .

From eqns (52), (54) and (55) we see that, asymptotically, the weak field solution describes the gravitational field about a central source of rest mass  $N$  so that to lowest order in  $\varepsilon$  the Tensor and Keplerian masses may be written as the products  $M_2 = G_2 N$ ,  $M_4 = G_4 N$  where the coupling strengths  $G_2$ ,  $G_4$  are given by

$$G_2 = e^{\phi_\infty} \quad G_4 = e^{\phi_\infty} \frac{2(\omega + 2)}{2\omega + 3}. \quad (57)$$

Hence the weak field solutions are Newtonian in the sense that the rest mass of the star may be separated from the gravitational coupling strength. However, this coupling is not unique and its strength depends upon the properties of the orbiting test particle that we use to measure the star's mass. The quantity  $G_4$ , often quoted as the effective gravitational constant in the weak field limit (see for example [18]), is the coupling strength one would measure in a Cavendish experiment using non-self gravitating test particles. Performing the same experiment using test black holes, one would measure the coupling strength  $G_2$ .

We use eqns (52), (53) and (55) to calculate the fractional binding energies associated with the ADM and Keplerian masses. The result is

$$E_0 = \frac{-1}{2\omega + 3} + \mathcal{O}(\varepsilon), \quad E_4 = \frac{1}{2\omega + 3} + \mathcal{O}(\varepsilon). \quad (58)$$

Both of these quantities are independent of  $\varepsilon$  in their leading term which is physically unreasonable: as  $\varepsilon \rightarrow 0$  and the solution approaches flat spacetime, we would expect the fractional binding energy to vanish as it does in the GR case

[20]. The fact that  $E_0$  and  $E_4$  behave in a non-physical way indicates that neither the ADM mass nor the Keplerian mass correctly describes the total energy of the star.

As is apparent when considering eqns (52) and (54), to calculate  $E_2$  we need to express  $M_N$  and  $M_T$  to order  $\varepsilon^{3/2}$ . It turns out that the  $\mathcal{O}(\varepsilon^{3/2})$  terms of both of these quantities depend only upon the  $\mathcal{O}(\varepsilon)$  weak field variables defined in eqn (47). One can then show that the fractional binding energy associated with the Tensor mass is given by

$$E_2 = -\frac{\varepsilon}{\mathcal{N}} \int_0^\infty \frac{a^3 \Pi^2}{2\omega + 3} \left[ \Phi'(\omega + 2) + B' \frac{(2\omega + 1)}{4} \right] da + \mathcal{O}(\varepsilon^2), \quad (59)$$

where we have defined

$$\mathcal{N} := \int_0^\infty \frac{a^2 \Pi^2}{2} da \quad (60)$$

so that  $\sqrt{\varepsilon} \mathcal{N}$  is the rest mass to lowest order in  $\varepsilon$ . We need to simplify eqn (59). Integrating eqns (48) and (50) one can show that

$$\Phi' = \frac{B'}{2(\omega + 2)} \quad (61)$$

and substituting this result into eqn (59) gives

$$E_2 = -\frac{\varepsilon}{\mathcal{N}} \int_0^\infty \frac{a^3 \Pi^2}{4} B' da + \mathcal{O}(\varepsilon^2). \quad (62)$$

Integrating both  $a^2 \Pi'^2$  and  $[a \Pi' (a^2 \Pi')']$  by parts and using eqn (51) one can show that

$$\int_0^\infty a^3 \Pi^2 B' da = -2 \int_0^\infty a^2 \Pi^2 (\Gamma + B) da. \quad (63)$$

Similarly, integrating both  $a^2 B'^2$  and  $[a B' (a^2 B')']$  by parts and using eqn (48) one can show that

$$\int_0^\infty a^2 \Pi^2 B da = -2 \int_0^\infty a^3 \Pi^2 B' da. \quad (64)$$

Substituting eqn (64) into (63) gives

$$\int_0^\infty a^3 \Pi^2 B' da = \frac{2\Gamma}{3} \int_0^\infty a^2 \Pi^2 da = \frac{2\Gamma}{3} \mathcal{N} \quad (65)$$

where we have used eqn (60) to obtain the last equality. Combining the above result with eqn (62) gives

$$E_2 = -\frac{\varepsilon \Gamma}{6} + \mathcal{O}(\varepsilon^2). \quad (66)$$

This is identical to the expression for the GR binding energy found by Kaup [20] and, in particular, is independent of  $\omega$  and vanishes as  $\varepsilon \rightarrow 0$ . Hence all weak field BD boson stars are energetically stable.

## VII. STRONG FIELD SOLUTIONS

We have numerically integrated the field equations (11) to (14) for the case  $\omega = -1$  over the parameter range  $0 \leq P_0 \leq 1.6$ . Results of the integration are given in Figure 1, which shows the behaviour of the three asymptotic masses we have discussed and how they relate to the Newtonian mass  $M_N$ . We have chosen the boundary value  $e^{\phi_\infty} = 1$  so that  $M_N$  is numerically equal to  $N$  and our results may be compared with those of other authors, although this choice is not physically realistic: by setting  $\omega = -1$  we are implicitly describing boson star solutions in the early (string-theory) Universe. In general, ST cosmological solutions show that the cosmological value of  $\phi$  decreases with cosmic time, so that we should choose the boundary value  $e^{\phi_\infty} \gg 1$ . However, from the scaling relation (42), changing the asymptotic value of  $\phi$  merely changes the scale of the horizontal axis of the figure (although

for  $e^{\phi_\infty} \neq 1$ ,  $M_N$  and  $N$  are no longer equal). Note also that the fractional binding energy of any solution is invariant under the rescaling of  $\phi_\infty$  (this result is true for all values of  $\omega$ ).

The four mass curves exhibit qualitatively similar behaviour to the ADM mass and particle number curves found for GR boson stars. Each varies smoothly with  $P_0$  and each mass curve reaches an initial maximum before falling off with increasing  $P_0$  and tending towards low amplitude oscillations about some low mass value. There are, however, several quantitative differences between the GR and BD solutions.

All of the solutions investigated possess a region in which  $-\phi''$  is large and negative and dominates over the boson field contribution to the total energy density. This implies that  $\rho < 0$  in this region, which leads to the condition  $m'_S < 0$  there, in violation of condition (c) given in Section III. The quasi-local Tensor and Keplerian masses  $m_T$  and  $m_K$  were found to be increasing functions of  $r$  for all solutions.

An obvious feature of the Figure is that the asymptotic masses  $M_{ADM}$ ,  $M_K$  and  $M_T$  differ remarkably from each other. This is a result of choosing such a low value of  $\omega$ , which means that the gradient of  $\phi$  is large, even in the asymptotic limit. Consequently, for these solutions  $\phi_1$  is of the same order of magnitude as  $M_{ADM}$  and from eqn (39) the difference between the values of the three masses at each  $P_0$  is large. As  $\omega$  increases, the dilaton field becomes less strongly coupled to the curvature and the ratio  $\phi_1/M_{ADM}$  decreases for a given  $N$ , which implies that both  $M_K$  and  $M_T$  tend towards  $M_{ADM}$ . From eqns (53) and (54), for a given  $P_0$  the fractional difference between  $M_{ADM}$  and  $M_T$  in the weak field limit is given by

$$\frac{M_T - M_{ADM}}{M_T} = \frac{1}{2\omega + 3}. \quad (67)$$

Further numerical work suggests that this relation is approximately true for strong field solutions, with an accuracy that increases with  $\omega$ . For the  $\omega = 6$  coupling chosen by Gunderson & Jensen [9], the fractional difference between  $M_{ADM}$  and  $M_T$  is around 1/15, which implies that the boson star masses that they quote are about 93% of their true values. For the choice  $\omega = 500$  implied by the current observational constraints, the fractional difference between the two masses is less than  $10^{-3}$  which, as pointed out in [15], is negligible.

A second obvious feature of the Figure is that the  $M_{ADM}$ ,  $M_K$  and  $M_T$  curves have extrema located at different values of  $P_0$ . This feature persists for all finite  $\omega$  and is only absent in the limit  $\omega \rightarrow \infty$  where the three masses are identical. However, it does appear from the figure that the extrema of  $M_T$  and  $M_N$  do occur at the same values of  $P_0$ , which suggests that  $M_T$  is the correct definition of the total energy of the star. In the following Section we give a proof that this is indeed the case for all values of  $\omega$ .

### VIII. COINCIDENCE OF EXTREMAL TENSOR MASS AND EXTREMAL PARTICLE NUMBER SOLUTIONS

We consider variations  $\delta M_T$  and  $\delta M_N$  in the Tensor and Newtonian masses induced by the parameter change  $P_0 \rightarrow P_0 + \delta P_0$  with  $\phi_\infty$  held fixed. Since each pair  $P_0, \phi_\infty$  gives a unique asymptotically flat zero node solution to the equations of structure (11) to (14), we may equivalently consider arbitrary variations in  $M_T$  and  $M_N$  subject to the constraint that the field equations and the boundary conditions (15) to (17) hold. We shall then end up with a simple relationship between  $\delta M_T$  and  $\delta M_N$ . Note that because of our choice of coordinates, the variation  $\delta$  commutes with both integration and differentiation with respect to  $r$ .

We start by taking the variation in the rest mass  $N$ . From the definition (19) we have

$$\delta N = \int_0^\infty \frac{r^2}{2} [e^{\lambda-\nu} \Omega P^2 \delta \lambda + e^\lambda \delta (e^{-\nu} \Omega P^2)] dr. \quad (68)$$

Using a similar method to the one outlined in [6] we rewrite the second term in the above integrand in the following way. One can easily show that

$$\delta (e^{-\nu} \Omega P^2) = \frac{1}{2\Omega} e^{\nu-\phi} \delta (e^{\phi-2\nu} \Omega^2 P^2) - e^{-\nu} \frac{\Omega P^2}{2} \delta \phi + e^{-\nu} \Omega P \delta P. \quad (69)$$

We solve eqn (24) for the term proportional to  $\Omega^2 P^2$  and substitute the result into the right hand side of the above equation. Then, substituting this result into eqn (68) gives

$$\begin{aligned} \delta N = \int_0^\infty \frac{r^2}{2\Omega} e^{\nu+\lambda-\phi} & [\delta \rho + e^{\phi-2\lambda} (e^{2\lambda-2\nu} \Omega^2 P^2 + P'^2 + (2+\omega) e^{-\phi} \phi'^2) \delta \lambda + e^\phi P (e^{-2\nu} \Omega^2 - 1) \delta P \\ & - \frac{e^\phi}{2} (e^{-2\nu} \Omega^2 P^2 + P^2 + e^{-2\lambda} P'^2) \delta \phi - e^{\phi-2\lambda} P' \delta P' - (2+\omega) e^{-2\lambda} \phi' \delta \phi' + \delta(\Delta \phi)] dr. \end{aligned} \quad (70)$$

Integrating the  $\delta P'$  term by parts and using the wave equation (14) one obtains an expression that cancels the  $\delta P$  term in eqn (70). Integrating the  $\delta(\Delta\phi)$  term by parts we have

$$\int_0^\infty \frac{r^2}{2\Omega} e^{\nu+\lambda-\phi} \delta(\Delta\phi) dr = \int_0^\infty \frac{r^2}{2\Omega} e^{\nu+\lambda-\phi} [(e^{-2\lambda}\nu'\phi' - e^{-2\lambda}\phi'^2 - \Delta\phi) \delta\lambda + e^{-2\lambda}(\phi' - \nu')\delta\phi'] dr + \lim_{r \rightarrow \infty} \left[ \frac{e^{\nu-\phi}}{2\Omega} \delta(e^{-\lambda}r^2\phi') \right]. \quad (71)$$

Substituting this result into eqn (70) gives

$$\delta N = \int_0^\infty \frac{r^2}{2\Omega} e^{\nu+\lambda-\phi} [\delta\rho + e^\phi (e^{-2\nu}\Omega^2 P^2 + e^{-2\lambda}P'^2 + (1+\omega)e^{-2\lambda-\phi}\phi'^2 - e^{-\phi}\Delta\phi + e^{-2\lambda-\phi}\nu'\phi') \delta\lambda - \frac{e^\phi}{2} (e^{-2\nu}\Omega^2 P^2 + P^2 + e^{-2\lambda}P'^2) \delta\phi - e^{-2\lambda}(\nu' + (1+\omega)\phi') \delta\phi'] dr + \lim_{r \rightarrow \infty} \left[ \frac{e^{\nu-\phi}}{2\Omega} \delta(e^{-\lambda}r^2\phi') \right]. \quad (72)$$

Integrating the  $\delta\phi'$  term by parts and substituting the result back into eqn (72) we have

$$\delta N = \int_0^\infty \frac{r^2}{2\Omega} e^{\nu+\lambda-\phi} [\delta\rho + e^\phi (e^{-2\nu}\Omega^2 P^2 + e^{-2\lambda}P'^2 + (1+\omega)e^{-2\lambda-\phi}\phi'^2 - e^{-\phi}\Delta\phi + e^{-2\lambda-\phi}\nu'\phi') \delta\lambda - (\rho + \frac{1}{2}\omega e^{-2\lambda}\phi'^2 - \square\nu - \omega\square\phi) \delta\phi] dr + \lim_{r \rightarrow \infty} \left[ \frac{e^{\nu-\phi}}{2\Omega} \delta(e^{-\lambda}r^2\phi') \right] \quad (73)$$

where we have used eqn (24) to rewrite part of the coefficient of  $\delta\phi$  in terms of the energy density  $\rho$  and rewritten some of the derivatives as the wave operator  $\square$ . This latter quantity has the explicit form

$$\square f := r^{-2} e^{-\nu-\lambda} \frac{d}{dr} (r^2 e^{\nu-\lambda} f') \quad (74)$$

for any function  $f(r)$ .

From the definition of the Einstein tensor we have

$$G_{\alpha\beta} \xi^\alpha \xi^\beta = R_{\alpha\beta} \xi^\alpha \xi^\beta + \frac{\mathcal{R}}{2} \quad (75)$$

where  $R_{\alpha\beta}$  is the Ricci tensor. Rearranging this equation, substituting in the explicit form of  $R_{\alpha\beta} \xi^\alpha \xi^\beta = e^{-2\nu} R_{00}$  and using the definition (23) we have

$$\frac{\mathcal{R}}{2} = \rho - \frac{1}{r^2} e^{-\nu-\lambda} \frac{d}{dr} (r^2 e^{\nu-\lambda} \nu') = \rho - \square\nu \quad (76)$$

where we have used eqn (74) to obtain the second equality. Solving eqn (4) for  $\mathcal{R}$ , substituting the result into the above equation and evaluating the  $\nabla^\sigma \phi \nabla_\sigma \phi$  term gives

$$\rho + \frac{1}{2}\omega e^{-2\lambda}\phi'^2 - \square\nu - \omega\square\phi = 0. \quad (77)$$

Hence the  $\delta\phi$  term in eqn (73) vanishes.

Combining eqns (11), (22), (23) and (24) one can show that

$$e^{2\lambda-2\nu}\Omega^2 P^2 + P'^2 - e^{2\lambda-\phi}\Delta\phi + e^{-\phi}\phi' [\nu' + (1+\omega)\phi'] = \frac{2}{r} e^{-\nu-\lambda} \frac{d}{dr} (e^{\nu+\lambda-\phi}) \quad (78)$$

Substituting this result into the remaining part of eqn (73) gives

$$\delta N = \int_0^\infty \frac{r^2}{2\Omega} \left[ e^{\nu-\lambda-\phi} \delta\rho + \frac{\delta\lambda}{r} e^{-2\lambda} \frac{d}{dr} (e^{\nu+\lambda-\phi}) \right] dr + \lim_{r \rightarrow \infty} \left[ \frac{e^{\nu-\phi}}{2\Omega} \delta(e^{-\lambda}r^2\phi') \right]. \quad (79)$$

Taking the variation of eqns (22) one can show that

$$\delta\rho = \frac{2}{r^2} \frac{d}{dr} (\delta m_S) \quad (80)$$

and

$$\delta\lambda = \frac{e^{2\lambda}}{r}\delta m_S. \quad (81)$$

Substituting both of these results into eqn (79) gives

$$\delta N = \int_0^\infty \frac{1}{\Omega} \frac{d}{dr} (e^{\nu+\lambda-\phi} \delta m_S) dr + \lim_{r \rightarrow \infty} \left[ \frac{e^{\nu-\phi}}{2\Omega} \delta (e^{-\lambda} r^2 \phi') \right]. \quad (82)$$

Evaluating the integral and using the boundary conditions (15) to (17) we have

$$\delta N = e^{-\phi_\infty} \lim_{r \rightarrow \infty} \left[ \frac{\delta m_S}{\Omega} + \frac{\delta(r^2 \phi')}{\Omega} \right] = \frac{1}{\Omega} e^{-\phi_\infty} \delta M_T \quad (83)$$

where we have used eqns (33), (38) and (39) to obtain the second equality. Finally, using the definition (20) we have

$$\delta M_T = \Omega \delta M_N. \quad (84)$$

The above equation gives the relationship between the variations in the Tensor and Newtonian masses under the parameter change  $P_0 \rightarrow P_0 + \delta P_0$  with  $\phi_\infty$  held fixed. Taking the limit  $\delta P_0 \rightarrow 0$  we have

$$\left. \frac{dM_T}{dP_0} \right|_{\phi_\infty} = \Omega \left. \frac{dM_N}{dP_0} \right|_{\phi_\infty} \quad (85)$$

Hence for any  $\omega$  and  $\phi_\infty$ , a solution that extremises the Newtonian mass is also one of extremal Tensor mass. In the GR limit,  $M_T \rightarrow M_{ADM}$  and we recover the result derived in [6].

## IX. CONCLUSIONS

We have examined three alternative definitions of mass in ST gravity: the ADM mass  $M_{ADM}$ , the Keplerian mass  $M_K$  and the Tensor mass  $M_T$ . In a static, spherically symmetric spacetime, all three may be written as the asymptotic limit of some quasi-local mass defined on the hypersurfaces of constant time. In the case of the ADM mass we have written this as the asymptotic limit of the Schwarzschild mass  $m_S$  which is defined in the usual way as the integral of the total energy density  $G_{\mu\nu}\xi^\mu\xi^\nu$  over a bounded subset of a constant time hypersurface. For a static spacetime in GR, the Schwarzschild mass defined in this way gives a perfectly satisfactory notion of quasi-local mass. We have analysed the behaviour of the asymptotic and quasi-local masses for BD boson stars in the strong coupling ( $\omega = -1$ ) case and found that both  $m_S$  and  $M_{ADM}$  have undesirable physical properties even though the boson field energy momentum tensor obeys the weak energy condition. In addition, for strong couplings,  $M_{ADM} < M_N$  for all values of the central boson field amplitude. If one assumes that the energy of the star is given by  $M_{ADM}$ , one would have to conclude that all stars in the strong coupling case are energetically stable, regardless of their central density. This contrasts strongly with the behaviour of boson star solutions in both GR and in weakly coupled BD theory. In all of these cases, one finds that the solutions become unstable above a certain value of the central density. These results imply that  $M_{ADM}$  does not give a correct description of the energy of the spacetime, primarily because it does not correctly include the dilaton field's contribution to the total energy. The dilaton, being an extra component of the gravitational field, must be treated on a slightly different footing to the normal matter fields.

Both  $M_T$  and  $M_K$  adequately include the contribution made by the dilaton to the active gravitational mass of the boson star as seen by two different kinds of orbiting test particles. However, as pointed out by Lee [8], out of the range of possible masses one can derive from the superpotential (30), it is only the Tensor mass that is conserved in a non-static isolated source. This implies that the Tensor mass may be the true physical energy of the star. However, the analysis given by Lee does not show that  $M_T$  uniquely satisfies a conservation law: one can conceive of other definitions of energy that are also conserved. In this work we have shown that, for a one parameter set of boson star solutions in BD theory,  $M_T$  is unique in that it is the only conserved quantity whose extremal points coincide with those of the boson particle number. In this respect,  $M_T$  is the analogue of the ADM energy used in GR and this behaviour supports the conjecture that  $M_T$  uniquely describes the energy of the system. We see no reason why these results should not hold for stellar objects composed of matter other than bosons, or in more general scalar-tensor theories.

For the weak coupling case, the numerical difference between  $M_T$  and  $M_{ADM}$  is small, so in practice there is little accuracy lost in identifying the energy of a boson star with  $M_{ADM}$ . However, in the early Universe and in more general scalar-tensor theories, one may have to consider strong couplings and in these cases it is important that the correct definition of mass is used.

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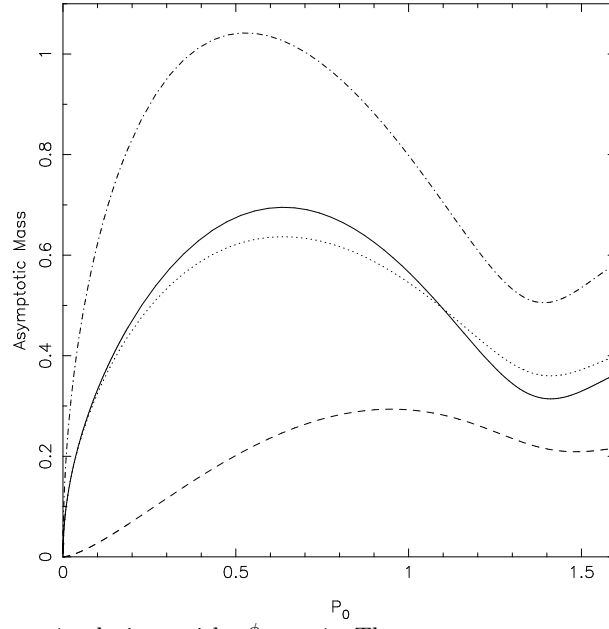


FIG. 1. Mass curves for the  $\omega = -1$  solutions with  $e^{\phi_\infty} = 1$ . The curves are parameterised by  $P_0$ , the amplitude of the boson wave function at the origin. Four mass curves are shown: the Newtonian mass  $M_N$  (solid line), the ADM mass  $M_{ADM}$  (dashed line), the Tensor mass  $M_T$  (dotted line) and the Keplerian mass  $M_K$  (dot-dashed line). All masses are expressed in units of  $M_{pl}^2/\alpha$ .